Using Exchangeable Pairs for Matrix Concentration Inequalities

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Based on work by Lester Mackey, Michael Jordan, Richard Chen, Brendan Farrell, and Joel Tropp

Matrix Concentration inequalities

Questions: given a random matrix X,

- $\mathbb{E}[\lambda_{max/min}(X)] = ??$

- $\Pr[\lambda_{max/min}$ far from $\mathbb{E}] = ??$

- Spectral norm ?

- Eigenvectors ?

- etc...

Matrix Concentration inequalities

Applications:

- Spectral graph theory
- randomized linear algebra
- Combinatorial & robust optimization
- stability of least-squares approximation
- .... and more!

matrix concentration inequalities

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* $\Pr[\lambda_{max}(X) \geq t] \leq ??$

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To Do:

- trace moment generating function

- matrix laplace inequalities

- exchangeable pairs — background + lemmas

- mean value trace inequality

- Bonus if $\exists$ time: example!

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$\begin{cases} X = \text{the conjugate transpose of } X \\ \text{all eigenvalues of } X \text{ are real} \end{cases}$

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matrix exponential: $e^{\theta X} = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} X^k$

normalized trace: $\bar{\mathrm{tr}}(A) = \frac{1}{d} \sum_{j=1}^{d} A_{jj}$

matrix laplace transform inequalities:

$X \in \mathcal{H}^d$ is a $d \times d$ random Hermitian matrix. Then for all $t \in \mathbb{R}$:

$$\Pr\left[\lambda_{max}(X) \geq t\right] \leq d \cdot \inf_{\theta > 0} e^{\wedge}\left(-\theta t + \log(m(\theta))\right)$$

$$\mathbb{E}\left[\lambda_{max}(X)\right] \leq \inf_{\theta > 0} \frac{1}{\theta}\left(\log(d) + \log(m(\theta))\right).$$

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Idea: bound $\log(m(\theta))$ to get concentration inequalities

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- $$\frac{d}{d\theta}\log(m(\theta)) = \frac{m'(\theta)}{m(\theta)} \rightsquigarrow$$ goal: Solve for a differential inequality of $m(\theta)$.

Bounding $\log(m(\theta))$:

2 main tools:

- method of exchangeable pairs
- mean value trace inequality

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$\log(m(\theta)) \leq$??

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Let Z, Z' be random variables

- Z and Z' are exchangeable if the distributions

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Let Z, Z' be random variables

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$$(Z, Z') \overset{d}{=} (Z', Z)$$

examples

1) Z, Z' are independently drawn from the same distribution

2) $Z = Z'$ always (completely dependent)

3) $Z' = \begin{cases} Z+1 & \text{w/ Prob } 1/2 \\ Z-1 & \text{w/ Prob } 1/2 \end{cases}$

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Let $X, X' \in \mathcal{H}^d$ be random $d \times d$ Hermitian matrices

X, X' are a matrix-stein pair w/ scale factor $\alpha \in (0,1]$ if:

1) X, X' are exchangeable

2) $\mathbb{E}[X - X' \mid X] = \alpha X$ (almost surely)

3) $\mathbb{E}[\|X\|^2] < \infty$

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3) ** $\mathbb{E}[\|X\|^2] < \infty$

* $\exists$ a more generalized version of this def

** not strictly necessary, but we'll assume this

Exchangeable pairs:

Let X, X' be a matrix-stein pair with scale factor α.

The conditional variance is:

$$\Delta_X = \frac{1}{2\alpha} \mathbb{E}\left[(X - X')^2 \mid X \right]$$

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We'll consider Δ_X to be bounded if

$$\Delta_X \preccurlyeq cX + v\underbrace{I}_{d \times d \text{ identity matrix}} \quad \text{for some constants } c, v$$

Method of Exchangeable Pairs:

Suppose X, X' are a matrix-stein pair with scale factor α.

Let $F : \mathcal{H}^d \to \mathcal{H}^d$ be a measurable function that satisfies:

$$\mathbb{E}\left[\| (X - X') F(X) \| \right] < \infty$$

Then

$$\mathbb{E}\left[X F(X) \right] = \frac{1}{2\alpha} \mathbb{E}\left[(X - X')(F(X) - F(X')) \right].$$

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* Corollary: $\mathbb{E}[\Delta_X] = \mathbb{E}\left[\frac{1}{2\alpha}\,\mathbb{E}\left[(X - X')^2 \mid X\right]\right] = \mathbb{E}[X^2]$

Mean value Trace inequality:

Let I be an interval of $\mathbb{R}$. Suppose $g: I \to \mathbb{R}$ is weakly increasing and $h: I \to \mathbb{R}$ has a convex derivative. Then for all matrices $A, B \in \mathcal{H}^d(I)$, it holds that:

$$\overline{\mathrm{tr}}\left[(g(A) - g(B)) \cdot (h(A) - h(B)) \right]$$
$$\leq \frac{1}{2} \overline{\mathrm{tr}} \left[(g(A) - g(B))(A-B)(h'(A) + h'(B)) \right]$$

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$$\leq \frac{1}{2}\, \overline{\mathrm{tr}}\left[\left(g(A) - g(B)\right)(A-B)\left(h'(A) + h'(B)\right)\right]$$

* where we evaluate $g(A)$ by decomposing $A = UDU^{-1}$ and applying the function g to the diagonal entries of D:

$$g(A) = U\, g(D)\, U^{-1}.$$

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recall: $m(\theta) = \mathbb{E}\left[\overline{\mathrm{tr}}\left(e^{\theta X} \right) \right]$

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Use method of Exchangeable pairs

$$= \frac{1}{2\alpha}\mathbb{E}\left[\bar{\mathrm{tr}}\left(\underbrace{(X-X')}_{g(x)=x}\underbrace{\left(e^{\theta X} - e^{\theta X'}\right)}_{h(x)=e^{\theta x}}\right)\right]$$

Use mean value trace inequality

goal: bound $m'(\theta)/m(\theta)$

$$\frac{1}{2\alpha}\mathbb{E}\left[\overline{\mathrm{tr}}\left((x-x')(e^{\theta x}-e^{\theta x'})\right)\right]$$

$$\leq \frac{1}{2\alpha}\mathbb{E}\left[\frac{1}{2}\overline{\mathrm{tr}}\left((x-x')^2(\theta e^{\theta x}+\theta e^{\theta x'})\right)\right]$$

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$$(x-x')^2 e^{\theta x} = (x'-x)^2 e^{\theta x'} = (x-x')^2 e^{\theta x'}$$

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recall: $\Delta x = \frac{1}{2\alpha}\mathbb{E}\left[(x-x')^2 \mid x\right]$

Then we can refactor:

goal: bound $m'(\theta)/m(\theta)$

$$m'(\theta) \le \frac{\theta}{2\alpha} \mathbb{E}\left[\bar{\mathrm{tr}}\left((x-x')^2 e^{\theta x} \right) \right]$$

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$$= \theta \, \mathbb{E}\left[\operatorname{tr}\left(\underbrace{\Delta_X}_{\text{use the fact that this is bounded :}} e^{\theta X} \right) \right]$$

$$\Delta_X \preceq cX + vI$$

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$$\leq \theta\,\mathbb{E}\left[\bar{\mathrm{tr}}\left(cXe^{\theta X} + vIe^{\theta X}\right)\right]$$

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$$\frac{d}{d\theta}\log(m(\theta)) = \frac{m'(\theta)}{m(\theta)} \le \frac{V\theta}{1-C\theta}$$

** for $0 \le \theta < 1/c$

goal: bound $\log(m(\theta))$

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$$= \frac{V\theta^2}{2(1-C\theta)}$$

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$$\leq d \cdot e^{\wedge}\left(\frac{-t^2}{2v+2ct}\right)$$

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$$= \sqrt{2V\log(d)} + c\log(d)$$

Theorem:

Let X, X' be a matrix-stein pair, and suppose $\exists$ constants c, v for which $\Delta_X \preccurlyeq cX + vI$.

Then for all $t \geq 0$,

$$\Pr\left[\lambda_{max}(X) \geq t\right] \leq d \cdot e^{\left(\frac{-t^2}{2v+2ct}\right)}$$

and

$$\mathbb{E}\left[\lambda_{max}(X)\right] \leq \sqrt{2v\log(d)} + c\log(d)$$

Final Q: how can this be useful?

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1) Find a good candidate X' to be an exchangeable pair with X

2) Bound the conditional variance of the pair :

$$\Delta_X = \frac{1}{2\alpha} \mathbb{E}\left[(X - X')^2 \mid X \right] \preccurlyeq cX + vI$$

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If you can find X' w/ small conditional variance w/ X, then you can get tighter concentration inequalities on λ_{max}.

<u>If we have time:</u> an example of an X, X' pair:

Let $Y_1, \ldots, Y_k \in \mathcal{H}^d$ be independent random $d \times d$ Hermitian matrices with $\mathbb{E}[Y_k] = 0$ and $\mathbb{E}[\|Y_k\|^2] < \infty \ \forall k$.

Let $X = \sum_{j=1}^{k} Y_j$.

Construct X' by choosing $J \in [k]$ uniformly at random and sampling Y_J', an independent copy of Y_J.

Then let $X' = Y_J' + \sum_{j \neq J} Y_j$.

<u>If we have time:</u> an example of an X, X' pair:

1) Check that X, X' are exchangeable. (✓)

2) Find the scale factor α:

$$
\begin{aligned}
\mathbb{E}[X - X' \mid X] &= \mathbb{E}[Y'_J - Y_J \mid X] \\
&= \frac{1}{K} \sum_{j=1}^{K} \mathbb{E}[Y_j - Y'_j \mid X] \\
&= \frac{1}{K} \sum_{j=1}^{K} Y_j = \underbrace{\frac{1}{K}} X
\end{aligned}
$$

so $\alpha = \frac{1}{K}$

If we have time: an example of an x, x' pair:

3) Compute Δ_x:

$$\Delta_x = \frac{k}{2} \cdot \mathbb{E}\left[(x - x')^2 \mid x\right]$$

$$= \frac{k}{2} \cdot \frac{1}{k} \cdot \sum_{j=1}^{k} \mathbb{E}\left[(y_j - y_j')^2 \mid x\right]$$

$$= \frac{1}{2} \sum_{j=1}^{k} \left(y_j^2 - y_j \mathbb{E}[y_j'] - \mathbb{E}[y_j'] y_j + \mathbb{E}[y_j']^2 \right)$$

$$= \frac{1}{2} \sum_{j=1}^{k} \underbrace{\left(y_j^2 + \mathbb{E}[y_j]^2 \right)}$$

So if we can control the size of individual y_j and $\mathbb{E}[y_j]$, we can bound Δ_x as well!

Thank You !

. . . . Questions ?